ANALYSIS OF THE FRF’S CURVE ACCURACY OF TDOF SYSTEM USING LINEAR SWEPT-SINE EXCITATION METHOD

Oleh:

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Abstrak


Kata kunci: kurva FRF, eksitasi linear swept-sine, Sistem 2-DK, magnitudo FRF, frekuensi resonansi, durasi eksitasi.

INTRODUCTION

Impact hammer and shaker excitation are the most commonly used excitation techniques in FRF (Frequency Response Function) measurement (Scheffer and Girdhar, 2004; Mobley, 1999; Schwarz and Richardson, 1999; McConnell, 1995). Using shaker’s excitations, generally there is much better control on the frequency ranges excited as well as the level of force applied to the tested system (Peres et.al, 2010; Cloutie et.al, 2009; Füllekrug et.al., 2008). There are various type of shaker’s excitations can be used, however, linear swept-sine excitation is the most common type of excitation in FRF measurement (Yanto and Abidin, 2012(a), 2012(b); Zhuge, 2009; Peeters et.al., 2008; Climent, 2007, Göge et.al., 2007; Pauwels et.al., 2006).

Linear swept-sine excitation is a sinusoidal excitation of which excitation frequencies change linearly within time (Orlando et.al, 2008; Gloth and Sinapsis, 2004(a), 2004(b); Baoliang and Xia, 2003; Haritos, 2002). Analysis of swept time (excitation’s duration) and modal parameter effects on the FRF’s magnitude error of SDOF (Single Degree Of Freedom) System using linear swept-sine excitation was explained by Yanto (2013). Nevertheless, application of this linear swept-sine excitation to other system as well as TDOF (Two Degree Of Freedom) should be explained too in order accuracy level of the FRF’s curve obtained by this method is known well. Here, accuracy level of the FRF’s curves established refers to error percentage of the obtained FRF’s magnitudes and resonant frequencies values to its theoretical values of the TDOF System.

METHODOLOGY

A TDOF System contains two mass $m_1$ and $m_2$, two stiffness $k_1$ and $k_2$, two dampers $c_1$ and $c_2$, and two excitation forces $f_1(t)$ and $f_2(t)$ as shown in Figure 1(a). Two coordinates $y_1$ and $y_2$ describe both mass relative positions to its references position.
Free body diagram of a TDOF System model in Figure 1(a) can be illustrated by Figure 1(b). There are two free body diagrams in Figure 1(b). The first is free body diagram of $m_1$ and the second is free body diagram of $m_2$. If motion equations of a TDOF System model in Figure 1(a) are derived for each mass (see free body diagram of a TDOF System model in Figure 1(b)), then obtained

\[ m_1\ddot{y}_1(t) + c_1\dot{y}_1(t) + c_2[\ddot{y}_1(t) - \ddot{y}_2(t)] + k_1y_1(t) + k_2[y_1(t) - y_2(t)] = f_1(t) \]  
\[ m_2\ddot{y}_2(t) - c_1[\dddot{y}_1(t) - \dddot{y}_2(t)] - k_2[y_1(t) - y_2(t)] = f_2(t) \]

In matrix form, both equations (Equation (1) and Equation (2)) can be expressed with

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}
\begin{bmatrix}
  \ddot{y}_1(t) \\  \ddot{y}_2(t)
\end{bmatrix}
+ \begin{bmatrix}
  c_1 + c_2 & -c_2 \\
  -c_2 & c_2
\end{bmatrix}
\begin{bmatrix}
  \dot{y}_1(t) \\  \dot{y}_2(t)
\end{bmatrix}
+ \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}
\begin{bmatrix}
  y_1(t) \\  y_2(t)
\end{bmatrix}
= \begin{bmatrix}
  f_1(t) \\  f_2(t)
\end{bmatrix}
\]  

Equation (3) can be written in form

\[
[m]\ddot{y} + [c]\dot{y} + [k]y = f(t)
\]  

By applying Laplace transformation to Equation (4) (Ogata, 2002, 1995), obtained
\[
[m]s^2\{y(s)\} - s\{y(0)\} - \{y(0)\} + [c]s\{y(s)\} - s\{y(0)\} + [k]\{y(s)\} = \{F(s)\}
\]
\[
\{y(0)\} = 0 \quad \text{and} \quad \dot{y}(0) = 0
\]
\[
[m]s^2 + [c]s + [k]\{y(s)\} = \{F(s)\}
\]

Thus, Equation (3) by applying Laplace Transformation is

\[
\begin{bmatrix}
  m_1 & 0 \\
  0 & m_2
\end{bmatrix}s^2 + \begin{bmatrix}
  c_1 + c_2 & -c_2 \\
  -c_2 & -k_2
\end{bmatrix}s + \begin{bmatrix}
  k_1 + k_2 & -k_2 \\
  -k_2 & k_2
\end{bmatrix}\begin{bmatrix}
  Y_1(s) \\
  Y_2(s)
\end{bmatrix} = \begin{bmatrix}
  F_1(s) \\
  F_2(s)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2) & -(c_2 s + k_2) \\
  -(c_2 s + k_2) & m_2 s^2 + c_2 s + k_2
\end{bmatrix}\begin{bmatrix}
  Y_1(s) \\
  Y_2(s)
\end{bmatrix} = \begin{bmatrix}
  F_1(s) \\
  F_2(s)
\end{bmatrix}
\]

Rewriting Equation (6), obtained

\[
\begin{bmatrix}
  Y_1(s) \\
  Y_2(s)
\end{bmatrix} = \begin{bmatrix}
  m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2) & -(c_2 s + k_2) \\
  -(c_2 s + k_2) & m_2 s^2 + c_2 s + k_2
\end{bmatrix}^{-1}\begin{bmatrix}
  F_1(s) \\
  F_2(s)
\end{bmatrix}
\]

To solve inverse of matrix in Equation (7), let

\[
[A] = \begin{bmatrix}
  m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2) & -(c_2 s + k_2) \\
  -(c_2 s + k_2) & m_2 s^2 + c_2 s + k_2
\end{bmatrix} = \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\]

The inverse of matrix A is given by

\[
[A]^{-1} = \frac{1}{\text{det}[A]}[C_{\alpha}] = \frac{1}{\text{det}[A]}\begin{bmatrix}
  a_{22} & -a_{12} \\
  -a_{21} & a_{11}
\end{bmatrix}
\]

Where \(C_{\alpha}\) is the cofactor of \(a_\alpha\) in \(\text{det}[A]\) (Kreyszig, 2006). Determinant of \(A\) is

\[
\text{det}[A] = \left[ \begin{array}{cc} m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2) & -(c_2 s + k_2) \\
 -(c_2 s + k_2) & m_2 s^2 + c_2 s + k_2 \end{array} \right]^2
\]

\[
= \left[ m_1 m_2 s^4 + [m_1 c_2 + m_2 (c_1 + c_2)]s^3 + [m_1 k_2 + c_2 c_2 + m_2 (k_1 + k_2)]s^2 \\
+ (c_1 k_2 + c_2 k_1)s + k_1 k_2 \right]^2
\]

Thereby, the inverse of matrix A can be determined as in Equation (11).

\[
[A]^{-1} = \begin{bmatrix}
  m_2 s^2 + c_2 s + k_2 & c_2 s + k_2 \\
  c_2 s + k_2 & m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2)
\end{bmatrix}
\]

Next, consider matrix G is the inverse of matrix A.

\[
[G] = [A]^{-1}
\]

The matrix G contains matrix elements \(G_{11}(s)\), \(G_{12}(s)\), \(G_{21}(s)\), and \(G_{22}(s)\). Hence, Equation (6) can be written by

\[
\begin{bmatrix}
  Y_1(s) \\
  Y_2(s)
\end{bmatrix} = \begin{bmatrix}
  G_{11}(s) & G_{12}(s) \\
  G_{21}(s) & G_{22}(s)
\end{bmatrix}\begin{bmatrix}
  F_1(s) \\
  F_2(s)
\end{bmatrix}
\]

Equation (13) can be illustrated with input-output relationship of the TDOF System as shown in Figure 2.
Furthermore, $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$, and $G_{22}(s)$ in Figure 2 are transfer functions of the TDOF System, where subscripts are coordinate of response and stimulus respectively. The transfer functions $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$, and $G_{22}(s)$ are expressed respectively with

$$G_{11}(s) = \frac{m_2 s^2 + c_2 s + k_2}{\left( m_1 m_2 s^4 + \left[ m_1 c_2 + m_2 (c_1 + c_2) \right] s^3 + \left[ m_1 k_2 + c_1 c_2 + m_2 (k_1 + k_2) \right] s^2 \right) + (c_1 k_2 + c_2 k_1) s + k_1 k_2} \quad \ldots (14)$$

$$G_{12}(s) = \frac{c_2 s + k_2}{\left( m_1 m_2 s^4 + \left[ m_1 c_2 + m_2 (c_1 + c_2) \right] s^3 + \left[ m_1 k_2 + c_1 c_2 + m_2 (k_1 + k_2) \right] s^2 \right) + (c_1 k_2 + c_2 k_1) s + k_1 k_2} \quad \ldots (15)$$

$$G_{22}(s) = \frac{m_1 s^2 + (c_1 + c_2) s + (k_1 + k_2)}{\left( m_1 m_2 s^4 + \left[ m_1 c_2 + m_2 (c_1 + c_2) \right] s^3 + \left[ m_1 k_2 + c_1 c_2 + m_2 (k_1 + k_2) \right] s^2 \right) + (c_1 k_2 + c_2 k_1) s + k_1 k_2} \quad \ldots (16)$$

### A. The theoretical FRF

The theoretical FRF of the TDOF System at certain span frequency, $H_{ij, theo}(f)$, can be determined by evaluating system at steady state condition ($s = j \omega = j 2 \pi f$).

$$H_{ij, theo}(f) = G_{ij}(j 2 \pi f) \quad \ldots (17)$$

### B. The FRF is obtained by the linear swept-sine excitation

The linear swept-sine excitation $u(t)$ has starting frequency $f_0$, ending frequency that equal to span frequency $f_s$, swept-time $T_s$, and amplitude $A$ (Yanto, 2013). The $u(t)$ is expressed with

$$u(t) = A \sin \left( \frac{\pi}{2} (f_0 - f_s) \frac{T_s^2}{T_s^2} + 2 \pi f_s t \right) \quad \ldots (18)$$

The FRF of the TDOF System using the linear swept-sine excitation, $H_{ij, lin}(f)$, is determined by

$$H_{ij, lin}(f) = \frac{Y(f)}{U_j(f)} \quad \ldots (19)$$
where,
\[ Y_i(f) = \mathcal{F}[y(t)] \quad \text{and} \quad U_j(f) = \mathcal{F}[u(t)]. \]
The \( \mathcal{F} \) symbol is Fourier transformation.
Subscript \( i \) is coordinate of response and
subscript \( j \) is coordinate of stimulus.

RESULTS AND DISCUSSION

Value of input parameters in analysis of accuracy of the FRF’s curves of the TDOF System that obtained by using the linear swept-sine excitation are shown in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 )</td>
<td>40</td>
<td>kg</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>10</td>
<td>kg</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>48</td>
<td>Ns/m</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>8</td>
<td>Ns/m</td>
</tr>
<tr>
<td>( k_1 )</td>
<td>80000</td>
<td>N/m</td>
</tr>
<tr>
<td>( k_2 )</td>
<td>40000</td>
<td>N/m</td>
</tr>
<tr>
<td>( t_r )</td>
<td>4, 8, and 16</td>
<td>s</td>
</tr>
</tbody>
</table>

The FRF’s curves of the TDOF System with
\( T_r = 4 \) s are shown in Figure 3, 4, and 5.

Figure 3. \( H_{11}(f) \) of the TDOF System with \( T_r = 4 \) s.
Figure 4. $H_{12}(f)$ or $H_{21}(f)$ of the TDOF System with $T_r = 4$ s.

Figure 5. $H_{22}(f)$ of the TDOF System with $T_r = 4$ s.

The FRF’s curves in Figure 3, 4, and 5 are shown the FRF’s magnitude $|H_i(f)|$ of the TDOF system. Accuracy of the FRF’s curves established refers to error percentage of the obtained FRF’s magnitude and resonant frequencies $f_r$ values to its theoretical values at peak of the FRF’s curves. The obtained FRF’s magnitude using the linear swept-sine excitation and $f_r$ values are compared to its theoretical values with 4 s, 8 s, and 16 s of $T_r$ are shown in Table 2. The error percentage of the obtained FRF’s magnitude and resonant...
frequencies \( f_r \) values to its theoretical values at peak of the FRF’s curves are shown in Table 3.

Table 2. The obtained FRF’s magnitudes using the linear swept-sine excitation and resonant frequencies values are compared to its theoretical values.

<table>
<thead>
<tr>
<th>FRF</th>
<th>Theoretical</th>
<th>At the first peak of FRF’s curve</th>
<th>At the second peak of FRF’s curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{11}(f) )</td>
<td></td>
<td>( f_{11} &lt; \text{Hz} &gt; )</td>
<td>( H \left( f_{11} \right) &lt; \text{m/N} &gt; )</td>
</tr>
<tr>
<td>( H_{12}(f) )</td>
<td>6.04</td>
<td>52.05 ( \times 10^{-5} )</td>
<td>11.87</td>
</tr>
<tr>
<td>( H_{21}(f) )</td>
<td>6.04</td>
<td>81.34 ( \times 10^{-5} )</td>
<td>11.87</td>
</tr>
<tr>
<td>( H_{22}(f) )</td>
<td>6.04</td>
<td>126.90 ( \times 10^{-5} )</td>
<td>11.87</td>
</tr>
</tbody>
</table>

Using the linear swept-sine excitation with \( T_s = 4 \) s

<table>
<thead>
<tr>
<th>FRF</th>
<th>Theoretical</th>
<th>At the first peak of FRF’s curve</th>
<th>At the second peak of FRF’s curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{11}(f) )</td>
<td></td>
<td>( f_{11} &lt; \text{Hz} &gt; )</td>
<td>( H \left( f_{11} \right) &lt; \text{m/N} &gt; )</td>
</tr>
<tr>
<td>( H_{12}(f) )</td>
<td>6.00</td>
<td>38.55 ( \times 10^{-5} )</td>
<td>12.00</td>
</tr>
<tr>
<td>( H_{21}(f) )</td>
<td>6.00</td>
<td>59.82 ( \times 10^{-5} )</td>
<td>11.75</td>
</tr>
<tr>
<td>( H_{22}(f) )</td>
<td>6.00</td>
<td>94.43 ( \times 10^{-5} )</td>
<td>11.75</td>
</tr>
</tbody>
</table>

Using the linear swept-sine excitation with \( T_s = 8 \) s

<table>
<thead>
<tr>
<th>FRF</th>
<th>Theoretical</th>
<th>At the first peak of FRF’s curve</th>
<th>At the second peak of FRF’s curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{11}(f) )</td>
<td></td>
<td>( f_{11} &lt; \text{Hz} &gt; )</td>
<td>( H \left( f_{11} \right) &lt; \text{m/N} &gt; )</td>
</tr>
<tr>
<td>( H_{12}(f) )</td>
<td>6.00</td>
<td>45.32 ( \times 10^{-5} )</td>
<td>11.88</td>
</tr>
<tr>
<td>( H_{21}(f) )</td>
<td>6.00</td>
<td>70.34 ( \times 10^{-5} )</td>
<td>11.88</td>
</tr>
<tr>
<td>( H_{22}(f) )</td>
<td>6.00</td>
<td>111.00 ( \times 10^{-5} )</td>
<td>11.88</td>
</tr>
</tbody>
</table>

Using the linear swept-sine excitation with \( T_s = 16 \) s

<table>
<thead>
<tr>
<th>FRF</th>
<th>Theoretical</th>
<th>At the first peak of FRF’s curve</th>
<th>At the second peak of FRF’s curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{11}(f) )</td>
<td></td>
<td>( f_{11} &lt; \text{Hz} &gt; )</td>
<td>( H \left( f_{11} \right) &lt; \text{m/N} &gt; )</td>
</tr>
<tr>
<td>( H_{12}(f) )</td>
<td>6.06</td>
<td>47.40 ( \times 10^{-5} )</td>
<td>11.88</td>
</tr>
<tr>
<td>( H_{21}(f) )</td>
<td>6.06</td>
<td>74.38 ( \times 10^{-5} )</td>
<td>11.88</td>
</tr>
<tr>
<td>( H_{22}(f) )</td>
<td>6.06</td>
<td>115.30 ( \times 10^{-5} )</td>
<td>11.88</td>
</tr>
</tbody>
</table>

Table 3. The error percentage of the obtained FRF’s magnitude using the linear swept-sine excitation and resonant frequencies values to its theoretical values at peak of the FRF’s curves.

<table>
<thead>
<tr>
<th>FRF</th>
<th>Using the linear swept-sine excitation with ( T_s = 4 ) s</th>
<th>Error at the first peak of FRF’s curve</th>
<th>Error at the second peak of FRF’s curve</th>
<th>Error of ( f_{11} )</th>
<th>Error of ( H \left( f_{11} \right) )</th>
<th>Error of ( f_{12} )</th>
<th>Error of ( H \left( f_{12} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{11}(f) )</td>
<td></td>
<td>( f_{11} &lt; % )</td>
<td>( H \left( f_{11} \right) &lt; % )</td>
<td>0.66</td>
<td>25.94</td>
<td>1.10</td>
<td>25.28</td>
</tr>
<tr>
<td>( H_{12}(f) )</td>
<td></td>
<td>( f_{12} &lt; % )</td>
<td>( H \left( f_{12} \right) &lt; % )</td>
<td>0.66</td>
<td>26.46</td>
<td>1.01</td>
<td>20.72</td>
</tr>
<tr>
<td>( H_{21}(f) )</td>
<td></td>
<td>( f_{21} &lt; % )</td>
<td>( H \left( f_{21} \right) &lt; % )</td>
<td>0.66</td>
<td>25.59</td>
<td>1.01</td>
<td>25.26</td>
</tr>
</tbody>
</table>

Using the linear swept-sine excitation with \( T_s = 8 \) s

<table>
<thead>
<tr>
<th>FRF</th>
<th>Using the linear swept-sine excitation with ( T_s = 8 ) s</th>
<th>Error at the first peak of FRF’s curve</th>
<th>Error at the second peak of FRF’s curve</th>
<th>Error of ( f_{11} )</th>
<th>Error of ( H \left( f_{11} \right) )</th>
<th>Error of ( f_{12} )</th>
<th>Error of ( H \left( f_{12} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{11}(f) )</td>
<td></td>
<td>( f_{11} &lt; % )</td>
<td>( H \left( f_{11} \right) &lt; % )</td>
<td>0.66</td>
<td>12.93</td>
<td>0.08</td>
<td>1.34</td>
</tr>
<tr>
<td>( H_{12}(f) )</td>
<td></td>
<td>( f_{12} &lt; % )</td>
<td>( H \left( f_{12} \right) &lt; % )</td>
<td>0.66</td>
<td>13.52</td>
<td>0.08</td>
<td>2.05</td>
</tr>
<tr>
<td>( H_{21}(f) )</td>
<td></td>
<td>( f_{21} &lt; % )</td>
<td>( H \left( f_{21} \right) &lt; % )</td>
<td>0.66</td>
<td>12.53</td>
<td>0.08</td>
<td>1.60</td>
</tr>
</tbody>
</table>

Using the linear swept-sine excitation with \( T_s = 16 \) s

<table>
<thead>
<tr>
<th>FRF</th>
<th>Using the linear swept-sine excitation with ( T_s = 16 ) s</th>
<th>Error at the first peak of FRF’s curve</th>
<th>Error at the second peak of FRF’s curve</th>
<th>Error of ( f_{11} )</th>
<th>Error of ( H \left( f_{11} \right) )</th>
<th>Error of ( f_{12} )</th>
<th>Error of ( H \left( f_{12} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{11}(f) )</td>
<td></td>
<td>( f_{11} &lt; % )</td>
<td>( H \left( f_{11} \right) &lt; % )</td>
<td>0.33</td>
<td>8.93</td>
<td>0.08</td>
<td>0.67</td>
</tr>
<tr>
<td>( H_{12}(f) )</td>
<td></td>
<td>( f_{12} &lt; % )</td>
<td>( H \left( f_{12} \right) &lt; % )</td>
<td>0.33</td>
<td>8.56</td>
<td>0.08</td>
<td>1.00</td>
</tr>
<tr>
<td>( H_{21}(f) )</td>
<td></td>
<td>( f_{21} &lt; % )</td>
<td>( H \left( f_{21} \right) &lt; % )</td>
<td>0.33</td>
<td>9.14</td>
<td>0.08</td>
<td>0.81</td>
</tr>
</tbody>
</table>
The error of resonant frequency at peak of the FRF’s curve of the obtained FRF using linear swept-sine excitation are less than 2 % for all $T_r$. In the meantime, the error of peak of the FRF’s curve of the obtained FRF using linear swept-sine excitation are decrease with increase swept-time $T_r$.

CONCLUSION

This paper has been presented an analysis of accuracy of the FRF’s curves obtained by using linear swept-sine excitation. From analysis has been done, it can be concluded that accuracy of the FRF’s curves of the TDOF System depends on the duration of linear swept-sine excitation (swept time) of which applied to system. If swept-time of the linear swept-sine excitation applied to the TDOF System is increased, then error percentage of the obtained FRF’s magnitudes and resonant frequencies of the TDOF System would be decrease.

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